PUBLISHED BY INSTITUTE OF PHYSICS PUBLISHING FOR SISSA/ISAS
RECEIVED: *August 27, 2003*
ACCEPTED: *October 6, 2003*

Parallelisable heterotic backgrounds

José Figueroa-O'Farrill

School of Mathematics, University of Edinburgh
Scotland, U.K.
E-mail: j.m.figueroa@ed.ac.uk

Teruhiko Kawano and Satoshi Yamaguchi

Department of Physics, University of Tokyo
Hongo, Tokyo 113-0033, Japan
E-mail: kawano@hep-th.phys.s.u-tokyo.ac.jp,
yamaguch@hep-th.phys.s.u-tokyo.ac.jp

ABSTRACT: We classify the simply-connected supersymmetric parallelisable backgrounds of heterotic supergravity. They are all given by parallelised Lie groups admitting a bi-invariant lorentzian metric. We find examples preserving 4, 8, 10, 12, 14 and 16 of the 16 supersymmetries.

KEYWORDS: Superstrings and Heterotic Strings, Superstring Vacua, Differential and Algebraic Geometry.

Contents

1. Introduction

In this note we present a classification of simply-connected supersymmetric parallelisable backgrounds of heterotic string theory; equivalently, supersymmetric parallelisable backgrounds up to local isometry. We work in the supergravity limit and construct all parallelisable backgrounds of ten-dimensional type I supergravity coupled to supersymmetric Yang–Mills. Parallelisable backgrounds have been studied recently in the context of type II string theory [1]–[3]. The interest in parallelisable backgrounds stems from the fact that string theory on such backgrounds should be exactly solvable. This is clear for type II string theory and also for the heterotic backgrounds without gauge fields, since as we will show the dilaton is linear and hence can be described by a Liouville theory [4], whereas the geometry is that of a parallelised Lie group and hence can be described by a WZW model [5]. For the backgrounds with gauge fields and hence a nonlinear dilaton, the nonlinearity is only a function of the null coordinate x^- and, as in the homogeneous plane waves of [6, 7], this also should be exactly solvable.

This paper is organised as follows. In section 2 we briefly set out the problem by defining the theory under consideration: the supergravity limit of heterotic string theory. Under the assumption of parallelisability we write down the equations of motion for bosonic backgrounds and the conditions for preservation of some supersymmetry. In section 3 we briefly review the parallelised geometries which will be the focus of this paper. In section 4 we derive some useful consequences of preservation of supersymmetry which underlie the bulk of the analysis. In particular we show that any supersymmetric background with trivial gauge fields must have a constant or linear dilaton. This suggests breaking the problem into two, depending on whether or not we turn on the gauge fields. In section 5 we classify the supersymmetric backgrounds with a linear (or constant) dilaton. This follows closely the analysis in [2, 3] for type II backgrounds. In section 6 we classify the supersymmetric backgrounds with nonlinear dilaton, hence with nontrivial gauge fields. Our results are displayed in a number of tables, particularly table 4 which summarises the results. We conclude in section 7 with some comments on the moduli space of parallelisable backgrounds. Finally, appendix A, includes our conventions for Clifford algebras and the Clifford action of differential forms on spinors.

2. Heterotic supergravity

The field theory limit of the heterotic string [8] is given by ten-dimensional $N{=}1$ supergravity [9] coupled to $N{=}1$ supersymmetric Yang–Mills [10]. This theory was constructed in [11] generalising the construction in [12] for the abelian theory. The bosonic field content consists of a ten-dimensional lorentzian metric g, a real scalar ϕ (the dilaton), a 3-form H and a gauge field A whose field-strength F can be thought of as the curvature two-form of a connection on a gauge bundle P with gauge group $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$.

In the string frame, the bosonic action on a spacetime M is given by

$$I = \int_M \mathrm{dvol}_g\, e^{-2\phi} \left(R + 4|d\phi|^2 - \frac{1}{2}|H|^2 - \frac{N}{2}|F|^2 \right), \tag{2.1}$$

where the norms $|-|^2$ are the natural (indefinite) norms induced from the metric:

$$|d\phi|^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \qquad |H|^2 = \frac{1}{6}H_{\mu\nu\rho}H^{\mu\nu\rho} \qquad |F|^2 = \frac{1}{2}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}\,,$$

and where Tr is an invariant metric on the Lie algebra $\mathfrak{g}$ of the gauge group.

The 3-form H is not necessarily closed, and instead one has

$$dH = \frac{N}{2}\,\mathrm{Tr}\,F \wedge F - 2\alpha'\,\mathrm{tr}\,\Omega \wedge \Omega\,,$$

where Ω is the curvature of the (unique) metric connection with torsion 3-form H. We will be specialising to parallelisable backgrounds; that is, those with $\Omega = 0$. For these backgrounds, this equation simplifies to

$$dH = \frac{N}{2}\,\mathrm{Tr}\,F \wedge F\,. \tag{2.2}$$

JHEP10(2003)012

For parallelisable backgrounds, the bosonic equations of motion also simplify, and one is left with

$$
\begin{aligned}
H_{\mu\nu\rho}\nabla^\rho\phi &= 0 && (2.3)\\
\nabla_\mu\nabla_\nu\phi &= \frac{N}{4}\operatorname{Tr} F_{\mu\rho}F_\nu{}^\rho && (2.4)\\
4|d\phi|^2 &= |H|^2+\frac{3N}{2}|F|^2 && (2.5)\\
\nabla_\nu\left(e^{-2\phi}F^{\nu\mu}\right)+\left[A_\nu, e^{-2\phi}F^{\nu\mu}\right] &= \frac{1}{2}H^{\mu\nu\rho}e^{-2\phi}F_{\nu\rho}\,. && (2.6)
\end{aligned}
$$

This last equation is the Yang-Mills equation in this geometry and simply says that the gauge covariant divergence of $e^{-2\phi}F$ vanishes.

The fermionic fields in the theory consist of a gravitino ψ, a dilatino λ and a gaugino χ. Let $S_\pm$ denote spinor bundles on M associated to the real sixteen-dimensional chiral spinor representations of $\mathrm{Spin}(1,9)$. Then ψ is a section of $T^*M\otimes S_+$, χ is a $\mathfrak{g}$-valued section of S_+ (strictly speaking a section of $\operatorname{ad}P\otimes S_+$, where $\operatorname{ad}P$ is the adjoint bundle of P) and λ is a section of S_-. In a bosonic background, the supersymmetric variations of these fields are given by

$$
\begin{aligned}
\delta\psi &= \hat\nabla\varepsilon\\
\delta\lambda &= \left(d\phi+\frac{1}{2}H\right)\varepsilon\\
\delta\chi &= \frac{1}{2\sqrt{2}}F\varepsilon\,, && (2.7)
\end{aligned}
$$

where ε is a section of S_+, $\hat\nabla$ is the spin connection associated to the metric connection with torsion 3-form H, and as usual, differential forms act on spinors via the Clifford action as described in appendix A which also contains our conventions concerning Clifford algebras and their representations.

For the parallelisable backgrounds which are the focus of this paper, $\hat\nabla$ is flat; hence there are no local obstructions to finding parallel sections. Since we will be interested only in local metrics — equivalently, in simply-connected spacetimes — there will not be any global obstructions either, whence we will not dwell on the first of the above three equations except to remark the following fact. Since a Killing spinor is parallel with respect to a connection $\hat\nabla$, it is uniquely determined by its value at any given point. In particular, a Killing spinor is nowhere-vanishing. We will use this fact implicitly in deriving relations arising from the existence of Killing spinors.

In this paper we shall be concerned with simply-connected parallelisable supersymmetric heterotic backgrounds (M,g,ϕ,H,F), where (M,g,H) is parallelisable and where (g,ϕ,H,F) satisfy the equations (2.2)–(2.6), and such that there exists at least one nonzero spinor ε for which the variations (2.7) vanish.

JHEP10(2003)012

3. Parallelisable geometries

Space	Torsion		Dilaton
AdS_3	$dH = 0$	$\|H\|^2 < 0$	constant
$\mathbb{R}^{1,n}$, $n \geq 0$	$H = 0$		unconstrained
$\mathbb{R}^n$, $n \geq 1$	$H = 0$		unconstrained
S^3	$dH = 0$	$\|H\|^2 > 0$	constant
S^7	$dH \neq 0$	$\|H\|^2 > 0$	constant
$\mathrm{SU}(3)$	$dH = 0$	$\|H\|^2 > 0$	constant
$\mathrm{CW}_{2n}(\boldsymbol{\lambda})$	$dH = 0$	$\|H\|^2 = 0$	$\phi(x^-)$

Table 1: Elementary simply-connected parallalisable geometries.

In this section we will briefly review the basic facts concerning parallelisable geometries. For a recent treatment see [2]. We will say that a pseudo-riemannian manifold (M, g) is *parallelisable* if it admits a flat metric connection with torsion.

It is possible to list all the simply-connected parallelisable lorentzian manifolds in any dimension. This uses the following three basic theorems. The first theorem, due to Élie Cartan and Schouten [13, 14] and to Wolf [15, 16], says that the irreducible simply-connected parallelisable riemannian manifolds are the following: the real line with the standard metric and vanishing torsion, a simply-connected compact simple Lie group with a bi-invariant metric and the parallelising torsion of Cartan–Schouten, or S^7 with the canonical round metric and the torsion coming from octonionic multiplication. The second theorem due to Wolf [15, 16] and Cahen and Parker [17] states that the indecomposable parallelisable lorentzian manifolds are precisely the Lie groups with bi-invariant lorentzian metric and parallelising torsion. The third result is the classification of simply-connected Lie groups admitting bi-invariant lorentzian metrics, which follows from the structure theorem of Medina and Revoy [18] (see also [19]) on indecomposable Lie algebras admitting invariant lorentzian metrics. The simply connected lorentzian Lie groups are given by $(\mathbb{R}, -dt^2)$, the universal cover of $\mathrm{SL}(2,\mathbb{R})$ (i.e., AdS_3) and a subclass of the Cahen-Wallach [20] spaces $\mathrm{CW}_{2n}(\boldsymbol{\lambda})$. For completeness we recall their definition.

Let $M = \mathbb{R}^2 \times \mathbb{R}^{2n-2}$ with coordinates $(x^+, x^-, \boldsymbol{x})$ and let $\langle -, - \rangle$ denote the euclidean inner product on $\mathbb{R}^{2n-2}$. Let $J : \mathbb{R}^{2n-2} \to \mathbb{R}^{2n-2}$ be an invertible skew-symmetric linear map and let ω be the corresponding 2-form. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > 0$ denote the skew-eigenvalues of J. The Lie group $\mathrm{CW}_{2n}(\boldsymbol{\lambda})$ is diffeomorphic to M with metric

$$g = 2dx^+dx^- - \langle J\boldsymbol{x}, J\boldsymbol{x} \rangle (dx^-)^2 + \langle d\boldsymbol{x}, d\boldsymbol{x} \rangle , \tag{3.1}$$

and parallelising torsion given by

$$H = dx^- \wedge \omega . \tag{3.2}$$

In summary, we can now list the ingredients out of which we can build all ten-dimensional parallelisable lorentzian geometries. For each one we also list properties concerning the dilaton ϕ and the torsion 3-form H. These results are summarised in table 1, whose last column follows from equation (2.3).

Indeed, in the case of a Lie group, that is, when $dH = 0$, equation (2.3) says that $d\phi$ must be central, when thought of as an element in the Lie algebra. Since AdS_3, S^3 and $\mathrm{SU}(3)$ are simple, their Lie algebras have no centre, whence $d\phi = 0$. In the case of an abelian group there are no conditions, and in the case of $\mathrm{CW}_{2n}(\boldsymbol{\lambda})$, the Lie algebra has

Spacetime	Spacetime
$\mathrm{AdS}_3 \times S^7$	$\mathrm{AdS}_3 \times S^3 \times S^3 \times \mathbb{R}$
$\mathrm{AdS}_3 \times S^3 \times \mathbb{R}^4$	$\mathrm{AdS}_3 \times \mathbb{R}^7$
$\mathbb{R}^{1,0} \times S^3 \times S^3 \times S^3$	$\mathbb{R}^{1,1} \times \mathrm{SU}(3)$
$\mathbb{R}^{1,2} \times S^7$	$\mathbb{R}^{1,3} \times S^3 \times S^3$
$\mathbb{R}^{1,6} \times S^3$	$\mathbb{R}^{1,9}$
$\mathrm{CW}_{10}(\boldsymbol{\lambda})$	$\mathrm{CW}_8(\boldsymbol{\lambda}) \times \mathbb{R}^2$
$\mathrm{CW}_6(\boldsymbol{\lambda}) \times S^3 \times \mathbb{R}$	$\mathrm{CW}_6(\boldsymbol{\lambda}) \times \mathbb{R}^4$
$\mathrm{CW}_4(\boldsymbol{\lambda}) \times S^3 \times S^3$	$\mathrm{CW}_4(\boldsymbol{\lambda}) \times S^3 \times \mathbb{R}^3$
$\mathrm{CW}_4(\boldsymbol{\lambda}) \times \mathbb{R}^6$	

Table 2: Ten-dimensional simply-connected parallelisable spacetimes.

a one-dimensional centre corresponding to ∂_+, whose dual one-form is dx^-. This means that $d\phi$ must be proportional to dx^-, whence ϕ can only depend on x^-. Finally for S^7, equation (2.3) says that $\star H \wedge d\phi = 0$, which implies that $d\phi = 0$. To see this, notice that the parallelised S^7 possesses a nearly parallel G_2 structure and the differential forms decompose into irreducible types under G_2. For example, the one-forms corresponding to the irreducible seven-dimensional irreducible representation $\mathfrak{m}$ of G_2 coming from the embedding $G_2 \subset \mathrm{SO}(7)$, whereas the two-forms decompose into $\mathfrak{g}_2 \oplus \mathfrak{m}$, where $\mathfrak{g}_2$ is the adjoint representation which is irreducible since G_2 is simple. Now, H and $\star H$ both are G_2-invariant and hence the map $\Omega^1(S^7) \to \Omega^2(S^7)$ defined by $\theta \mapsto \star(\star H \wedge \theta)$ is G_2-equivariant. Since it is not identically zero, it must be an isomorphism onto its image. Hence if $\star H \wedge d\phi = 0$, then also in this case $d\phi = 0$.

It is now a simple matter to put these ingredients together to make up all possible ten-dimensional combinations with lorentzian signature. Doing so, we arrive at table 2 (see also [2], where the entry corresponding to $\mathbb{R}^{1,0} \times S^3 \times S^3 \times S^3$ had been omitted inadvertently and where the entries with S^7 had also been omitted due to the fact that in type II string theory $dH = 0$).

4. Some consequences of supersymmetry

In this section we will derive some consequences of the existence of nonzero Killing spinors.

The gaugino variation says that

$$F\varepsilon = 0 \ .$$

Clifford multiplying this equation by F and tracing over $\mathfrak{g}$, we find

$$\mathrm{Tr}(F \wedge F)\varepsilon = |F|^2\varepsilon \ . \tag{4.1}$$

At the same time, F is a $\mathfrak{g}$-valued 2-form. At a fixed but arbitrary point in M, F defines an element in $\mathfrak{so}(1,9) \otimes \mathfrak{g}$. Moreover, because F annihilates a nonzero spinor, it actually defines an element in $\mathfrak{h} \otimes \mathfrak{g}$, where $\mathfrak{h} \subset \mathfrak{so}(1,9)$ is the isotropy algebra of the spinor. Now the orbit structure of the chiral spinor representation under $\mathrm{Spin}(1,9)$ is very simple: all nonzero spinors belong to the same orbit [21]. The isotropy of a nonzero spinor is therefore conjugate in $\mathrm{Spin}(1,9)$ to a fixed $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ subgroup, which is described in detail, for example, in [22]. In other words, there exists a coframe $\boldsymbol{\theta} = (\theta^-, \theta^+, \theta^i)$ relative to which the metric is written as

$$g = 2\theta^+\theta^- + \sum_i (\theta^i)^2 \,,$$

and F is written as

$$F = \mathring{F}_{-i}\theta^- \wedge \theta^i + \frac{1}{2}\mathring{F}_{ij}\theta^i \wedge \theta^j \,,$$

where the $\mathring{F}_{ij}$ belong to a $\mathfrak{spin}(7)$ subalgebra of $\mathfrak{so}(8)$.

The norm $|F|^2$ is independent on the coframe, so we compute it relative to $\boldsymbol{\theta}$ and obtain

$$|F|^2 = \mathrm{Tr} \sum_{i<j} (\mathring{F}_{ij})^2 \geq 0 \,, \tag{4.2}$$

which is positive semidefinite. Similarly, rewriting equation (2.4) in this coframe, we find

$$\mathring{\nabla}_a \mathring{\nabla}_b \phi = \frac{N}{4} \,\mathrm{Tr} \sum_i \mathring{F}_{ai} \mathring{F}_{bi} \,,$$

which is also positive-definite; whence in this coframe, and hence in all, the Hessian of ϕ vanishes if and only if $F = 0$. In other words, supersymmetric backgrounds with $F = 0$ are precisely those with a linear dilaton.

Finally let us remark that if $dH = 0$, whence $\mathrm{Tr}(F \wedge F) = 0$, equation (4.1) implies that $|F|^2 = 0$, which by (4.2) implies that $\mathring{F}_{ij} = 0$. In other words, relative to the coframe $\boldsymbol{\theta}$ the only nonzero components of F are $\mathring{F}_{-i}$, so that

$$F_{\mu\nu} = \left(\theta_\mu{}^- \theta_\nu{}^i - \theta_\mu{}^i \theta_\nu{}^-\right) \mathring{F}_{-i} \,. \tag{4.3}$$

5. Linear dilaton backgrounds

In this section we consider (supersymmetric) backgrounds with $F = 0$. As we saw above, these are precisely the backgrounds where the hessian of the dilaton vanishes; whence the dilaton is (at most) linear.

First of all we notice that S^7 cannot appear because equation (2.2) implies that $dH = 0$. Therefore the allowed backgrounds follow *mutatis mutandis* from the analysis of [2, 3]. We only have to remember when counting supersymmetries that we are dealing with $N{=}1$ supergravity. In practice this simply means halving the supersymmetries present in type IIB supergravity. We start by listing the possible backgrounds and then counting the amount of supersymmetry that each preserves. The results are summarised in table 3 and table 4.

Geometry	Dilaton
$\mathrm{AdS}_3 \times S^3 \times S^3 \times \mathbb{R}$	$\phi = a + \frac{1}{2}\lvert H\rvert y$
$\mathrm{AdS}_3 \times S^3 \times \mathbb{R}^4$	$\phi = a + \frac{1}{2}\lvert H\rvert y$
$\mathbb{R}^{1,1} \times \mathrm{SU}(3)$	$\phi = a + \frac{1}{2}\lvert H\rvert y$
$\mathbb{R}^{1,3} \times S^3 \times S^3$	$\phi = a + \frac{1}{2}\lvert H\rvert y$
$\mathbb{R}^{1,6} \times S^3$	$\phi = a + \frac{1}{2}\lvert H\rvert y$
$\mathbb{R}^{1,9}$	$\phi = a + bx^-$
$\mathrm{CW}_{10}(\boldsymbol{\lambda})$	$\phi = a + bx^-$
$\mathrm{CW}_8(\boldsymbol{\lambda}) \times \mathbb{R}^2$	$\phi = a + bx^-$
$\mathrm{CW}_6(\boldsymbol{\lambda}) \times S^3 \times \mathbb{R}$	$\phi = a + bx^- + \frac{1}{2}\lvert H\rvert y$
$\mathrm{CW}_6(\boldsymbol{\lambda}) \times \mathbb{R}^4$	$\phi = a + bx^-$
$\mathrm{CW}_4(\boldsymbol{\lambda}) \times S^3 \times \mathbb{R}^3$	$\phi = a + bx^- + \frac{1}{2}\lvert H\rvert y$
$\mathrm{CW}_4(\boldsymbol{\lambda}) \times \mathbb{R}^6$	$\phi = a + bx^-$

Table 3: Parallelisable backgrounds with a linear dilaton. The notation is such that y is a spacelike flat coordinate.

Parallelisable geometry	Supersymmetries with dilaton being constant	linear	nonlinear
$\mathrm{AdS}_3 \times S^3 \times S^3 \times \mathbb{R}$	8	8	$\times$
$\mathrm{AdS}_3 \times S^3 \times \mathbb{R}^4$	8	8	$\times$
$\mathbb{R}^{1,1} \times \mathrm{SU}(3)$	$\times$	8	$\times$
$\mathbb{R}^{1,3} \times S^3 \times S^3$	$\times$	8	4
$\mathbb{R}^{1,6} \times S^3$	$\times$	8	4
$\mathbb{R}^{1,9}$	16	8	8
$\mathrm{CW}_{10}(\boldsymbol{\lambda})$	$8, 10, 12, 14$	8	8
$\mathrm{CW}_8(\boldsymbol{\lambda}) \times \mathbb{R}^2$	$8, 10$	8	8
$\mathrm{CW}_6(\boldsymbol{\lambda}) \times S^3 \times \mathbb{R}$	$\times$	8	4
$\mathrm{CW}_6(\boldsymbol{\lambda}) \times \mathbb{R}^4$	$8, 12$	8	8
$\mathrm{CW}_4(\boldsymbol{\lambda}) \times S^3 \times \mathbb{R}^3$	$\times$	8	4
$\mathrm{CW}_4(\boldsymbol{\lambda}) \times \mathbb{R}^6$	8	8	8

Table 4: Supersymmetric parallelisable backgrounds.

5.1 Possible backgrounds

1. $\mathrm{AdS}_3 \times S^3 \times S^3 \times \mathbb{R}$. Here $d\phi$ can only have nonzero components along the flat direction, which is spacelike, whence $|d\phi|^2 \geq 0$. Equation (2.5) says that $|H|^2 \geq 0$, so that if we call R_0, R_1 and R_2 the radii of curvature of AdS_3 and of the two 3-spheres, respectively, then
$$\frac{1}{R_1^2} + \frac{1}{R_2^2} \geq \frac{1}{R_0^2}\,.$$
This bound is saturated if and only if the dilaton is constant.

2. $\mathrm{AdS}_3 \times S^3 \times \mathbb{R}^4$. This is the limit $R_2 \to \infty$ of the above case.

3. $\mathrm{AdS}_3 \times \mathbb{R}^7$. This would be the limit $R_1 \to \infty$ of the above case, but then the inequality $R_0^{-2} \leq 0$ cannot be satisfied. Hence this geometry is not a background (with or without supersymmetry).

4. $\mathbb{R}^{1,9}$. In this case $H = 0$, so $|d\phi|^2 = 0$. So we can take a linear dilaton along a null direction: $\phi = a + bx^-$, for some constants a, b say.

5. $\mathbb{R}^{1,0} \times S^3 \times S^3 \times S^3$. The dilaton can only depend on the flat coordinate, which is timelike, so $|d\phi|^2 \leq 0$. However $|H|^2 > 0$, whence this geometry is never a background (with or without supersymmetry).

6. $\mathbb{R}^{1,1} \times \mathrm{SU}(3)$. Here $|H|^2 > 0$, and $d\phi$ can have components along $\mathbb{R}^{1,1}$. Letting (x^0, x^1) be flat coordinates for $\mathbb{R}^{1,1}$, we can take $\phi = a + \frac{1}{2}|H|x^1$, for some constant a, without loss of generality.

7. $\mathbb{R}^{1,3} \times S^3 \times S^3$. Here $|H|^2 > 0$ and $d\phi$ can have components along $\mathbb{R}^{1,3}$ [23]. With (x^0, x^1, x^2, x^3) being flat coordinates for $\mathbb{R}^{1,3}$, we take $\phi = a + \frac{1}{2}|H|x^1$, for some constant a.

8. $\mathbb{R}^{1,6} \times S^3$. This is the limit $R_2 \to \infty$ of the above case, where R_2 is the radius of curvature of one of the spheres [24, 4].

9. $\mathrm{CW}_{2n}(\boldsymbol{\lambda}) \times \mathbb{R}^{10-2n}$, $n = 2, 3, 4, 5$. In these cases $|H|^2 = 0$ and hence $|d\phi|^2 = 0$, so that it cannot have components along the flat directions (if any). This means $\phi = a + bx^-$, for constants a, b.

10. $\mathrm{CW}_4(\boldsymbol{\lambda}) \times S^3 \times S^3$. Here $|d\phi|^2 = 0$, whereas $|H|^2 > 0$, hence there are no backgrounds with this geometry.

11. $\mathrm{CW}_{2n}(\boldsymbol{\lambda}) \times S^3 \times \mathbb{R}^{7-2n}$, $n = 2, 3$. Here $|H|^2 > 0$, whence $|d\phi|^2 > 0$. This means that we can take $\phi = a + bx^- + \frac{1}{2}|H|y$, where y is any flat coordinate in $\mathbb{R}^{7-2n}$ and a, b are constants.

5.2 Supersymmetry

We must distinguish between three cases: $d\phi = 0$, $d\phi \neq 0$ but $|d\phi|^2 = 0$ and $|d\phi|^2 > 0$. The results are summarised in table 4.

1. $d\phi = 0$. This can be read off from [2] and we will not repeat the analysis here. We simply read off the results for type IIB and halve the number of supersymmetries.

2. $d\phi \neq 0$ and $|d\phi|^2 = 0$. This follows from [3]. Notice that this only occurs with $\mathrm{CW}_{2n}(\boldsymbol{\lambda}) \times \mathbb{R}^{10-2n}$ for $n = 2, 3, 4, 5$ and for $\mathbb{R}^{1,9}$. In the former cases, the dilatino variation implies an equation of the form
$$\left(b + \frac{1}{2}\omega\right) \cdot dx^- \varepsilon = 0\,,$$
where the $\cdot$ means Clifford multiplication as defined in appendix A. Clearly half the supersymmetries will be killed by dx^-, whereas the operator $b + \frac{1}{2}\omega$ is invertible since b is real and ω is invertible and has no real eigenvalues. In the latter case, the dilaton

variation is simply

$$dx^{-}\varepsilon = 0\,,$$

whence the background is also half-BPS.

3. $|d\phi|^2 > 0$. This also follows from [3]. Multiply the dilatino variation by $d\phi$ and using that $d\phi \cdot H = -H \cdot d\phi$, which follows from equation (2.3), to obtain

$$\left(-|d\phi|^2 \mathbb{1} + \frac{1}{2} d\phi \cdot H\right)\varepsilon = 0$$

which is equivalent to

$$\left(\frac{1}{2}\mathbb{1} + \frac{1}{4}\frac{H \cdot d\phi}{|d\phi|^2}\right)\varepsilon = 0\,.$$

Define endomorphisms

$$\Pi_{\pm} = \frac{1}{2}\mathbb{1} \pm \frac{1}{4}\frac{H \cdot d\phi}{|d\phi|^2}\,.$$

One sees immediately that $\Pi_+ + \Pi_- = \mathbb{1}$ and from equation (2.5) that $\Pi_+ \cdot \Pi_- = 0$, whence they are idempotent: $\Pi_{\pm}^2 = \Pi_{\pm}$. In other words, they are complementary projectors and their kernels are therefore half-dimensional. Therefore these backgrounds are also half-BPS.

6. Turning on the gauge fields

We now relax the condition that $F = 0$ and consider backgrounds with non-linear dilatons. We will go through each geometry in turn and determine which ones can carry a nontrivial gauge field. For those backgrounds we will then determine the amount of supersymmetry that is preserved. The results are summarised in table 4.

6.1 Possible backgrounds

1. $\mathrm{AdS}_3 \times S^7$. In this case the dilaton is forced to be constant, whence $F = 0$, which contradicts $dH \neq 0$.

2. $\mathrm{AdS}_3 \times X$, $X \neq S^7$. Chose coordinates $x^\mu = (x^0, x^m)$ where x^0 is timelike and x^m are spacelike. The timelike component of the gradient of the dilaton must vanish: $\nabla_0 \phi = 0$. Equation (2.4) then says that

$$\operatorname{Tr} F_{0m} {F_0}^m = 0 \implies F_{0m} = 0\,.$$

Since $dH = 0$, $|F|^2 = 0$, which implies that $F_{mn} = 0$. Therefore $F = 0$ and the dilaton has to be linear.

3. $\mathbb{R}^{1,0} \times S^3 \times S^3 \times S^3$. Here $|d\phi|^2 \leq 0$, whereas $|H|^2 > 0$ and $|F|^2 \geq 0$. Therefore equation (2.5) cannot be satisfied with or without supersymmetry.

JHEP10(2003)012

4. $\mathbb{R}^{1,1} \times \mathrm{SU}(3)$. Let us choose coordinates (x^+, x^-) for $\mathbb{R}^{1,1}$ and x^α for $\mathrm{SU}(3)$. It follows from (2.3) that $\nabla_\alpha \phi = 0$. Because $dH = 0$, equation (4.3) applies:

$$F_{\mu\nu} = \left(\theta_\mu{}^- \theta_\nu{}^i - \theta_\mu{}^i \theta_\nu{}^-\right) \mathring{F}_{-i}\,,$$

whence by (2.4),

$$\nabla_\mu \nabla_\nu \phi = \frac{N}{4} \theta_\mu{}^- \theta_\nu{}^- \operatorname{Tr} \sum_i (\mathring{F}_{-i})^2\,.$$

Since $\nabla_\alpha \phi = 0$, it follows that $\nabla_\alpha \nabla_\alpha \phi = 0$, whence

$$(\theta_\alpha{}^-)^2 \operatorname{Tr} \sum_i (\mathring{F}_{-i})^2 = 0\,.$$

Since $F \neq 0$, we must have $\theta_\alpha{}^- = 0$, whence $F_{\alpha\beta} = 0$; and since $|F|^2 = 0$, this yields the identity

$$\operatorname{Tr} F_{+-}^2 = 2 \operatorname{Tr} F_{+\alpha} F_-{}^\alpha\,. \tag{6.1}$$

Now since $\boldsymbol{\theta}$ is an orthonormal coframe, $\theta_\mu{}^- \theta_\nu{}^- g^{\mu\nu} = 0$. Since $\theta_\alpha{}^- = 0$, this implies that $\theta_+{}^- \theta_-{}^- = 0$, whence either $\theta_+{}^- = 0$ and hence $F_{+\alpha} = 0$ or else $\theta_-{}^- = 0$ and hence $F_{-\alpha} = 0$. In either case, equation (6.1) implies that $F_{+-} = 0$.

Without loss of generality, let us assume that $\theta_+{}^- = 0$, so that the only surviving components of F are $F_{-\alpha}$. This implies that the only surviving component of the hessian of ϕ is

$$\nabla_- \nabla_- \phi = \frac{N}{4} \operatorname{Tr} F_{-\alpha} F_-{}^\alpha\,.$$

Finally we use that equation (2.5) and the fact that $|H|^2$ is constant imply that so is $\nabla_+ \phi \nabla_- \phi$. Differentiating with respect to ∇_-, we obtain

$$\nabla_+ \phi \nabla_- \nabla_- \phi = 0\,,$$

whence either $\nabla_+ \phi = 0$, in which case $|d\phi|^2 = 0$ contradicting that $|H|^2 > 0$ and hence our assumption that $F \neq 0$, or else the hessian of ϕ vanishes, whence $F = 0$. In either case, we conclude that this geometry does not admit nontrivial gauge fields.

5. $\mathrm{CW}_4(\boldsymbol{\lambda}) \times S^3 \times S^3$. Here $|d\phi|^2 = 0$, whereas $|H|^2 > 0$ and $|F|^2 \geq 0$, so that equation (2.5) cannot be satisfied. This argument is independent on supersymmetry, whence this is not a parallelisable heterotic background with or without supersymmetry.

6. $\mathrm{CW}_{2n}(\boldsymbol{\lambda}) \times S^3 \times \mathbb{R}^{7-2n}$, $n = 2, 3$. These cases can be analysed simultaneously. We choose standard coordinates (x^+, x^-, x^m) for $\mathrm{CW}_{2n}(\boldsymbol{\lambda})$, x^α for S^3 and flat coordinates x^I for $\mathbb{R}^{7-2n}$.

   The only components of $\nabla \phi$ which are allowed to be nonzero are $\nabla_- \phi$ and $\nabla_I \phi$, where $\nabla_I \phi$ cannot all vanish because $|d\phi|^2 > 0$ since $|H|^2 > 0$.

Since $dH = 0$, equation (4.3) holds and substituting it into equation (2.4) we obtain

$$\nabla_\mu \nabla_\nu \phi = \frac{N}{4} \theta_\mu{}^- \theta_\nu{}^- \operatorname{Tr} \sum_i (\mathring{F}_{-i})^2 \,.$$

Since we are interested in $F \neq 0$, $\operatorname{Tr} \sum_i (\mathring{F}_{-i})^2 > 0$. This implies, together with $\nabla_+ \phi = \nabla_m \phi = \nabla_\alpha \phi = 0$, that $\theta_+{}^- = \theta_\alpha{}^- = \theta_m{}^- = 0$. Since $\boldsymbol{\theta}$ is orthonormal, $\theta_\mu{}^- \theta_\nu{}^- g^{\mu\nu} = 0$, whence $\theta_I{}^- = 0$, whence the only nonzero entry is $\theta_-{}^-$, which into (4.3) says that the only nonzero components of F are $F_{-\mu}$. Moreover, since $|F|^2 = 0$, it follows in addition that $F_{-+} = 0$.

This allows us to choose the gauge $A = A_- dx^-$, where A_- does not depend on x^+. It also means that the only nonzero component of the hessian of ϕ is

$$\nabla_- \nabla_- \phi = \frac{N}{4} g^{ij} \operatorname{Tr} F_{-i} F_{-j} \,,$$

where the indices i, j label collectively all the transverse coordinates (x^m, x^α, x^I). In particular, we see that $\nabla_- \nabla_I \phi = \nabla_I \nabla_J \phi = 0$. From this we see that

$$\phi = \varphi(x^-) + Q_I x^I \,,$$

where Q_I are constants obeying

$$\sum_I (Q_I)^2 = \frac{1}{4} |H|^2$$

so that equation (2.5) is satisfied; and $\varphi(x^-)$ satisfies the second order ODE:

$$\varphi'' = \frac{N}{4} g^{ij} \operatorname{Tr} \partial_i A_- \partial_j A_- \,, \tag{6.2}$$

which has a unique solution for specified initial conditions at least locally.

It remains to satisfy equation (2.6), which in this case simplifies to

$$g^{\nu\rho} \nabla_\nu \left(e^{-2\phi} F_{\rho\mu} \right) = 0 \implies g^{ij} \nabla_i \partial_j A_- = 2 \sum_I Q_I \partial_I A_- \,. \tag{6.3}$$

We now apply the transverse laplacian $\Delta^\perp = g^{ij} \nabla_i \nabla_j$ to (6.2) to obtain

$$|\nabla \partial A_-|^2 + R^{ij} \operatorname{Tr}(\partial_i A_- \partial_j A_-) + g^{ij} \operatorname{Tr} \nabla_i \Delta^\perp A_- \nabla_j A_- = 0 \,, \tag{6.4}$$

where $\nabla \partial A_-$ stands for the transverse hessian $\nabla_i \partial_j A_-$ of A_- and R^{ij} is the Ricci tensor. After using (6.3), this equation becomes

$$|\nabla \partial A_-|^2 + R^{ij} \operatorname{Tr}(\partial_i A_- \partial_j A_-) = - \sum_I Q_I \partial_I g^{ij} \operatorname{Tr} \partial_i A_- \partial_j A_- \,.$$

Now the right-hand side of this equation is zero because $g^{ij} \operatorname{Tr} \partial_i A_- \partial_j A_-$ is proportional to φ'' which only depends on x^-, whence we are left with

$$|\nabla \partial A_-|^2 + R^{ij} \operatorname{Tr}(\partial_i A_- \partial_j A_-) = 0 \,.$$

Now the Ricci tensor R^{ij} is only nonzero on the sphere, on which it is positive-definite, hence the above expression becomes

$$|\nabla\partial A_-|^2 + R^{\alpha\beta}\,\mathrm{Tr}(\partial_\alpha A_- \partial_\alpha A_-) = 0\,,$$

which is manifestly positive-definite and hence will vanish if and only if

$$\nabla_i\partial_j A_- = 0 \qquad \text{and} \qquad \partial_\alpha A_- = 0\,,$$

with the consequence that

$$A_- = a(x^-) + f_I(x^-)x^I\,.$$

Finally, in order to satisfy (6.3), we must demand $\sum_I Q_I f_I(x^-) = 0$.

7. $\mathrm{CW}_{2n}(\boldsymbol{\lambda}) \times \mathbb{R}^{10-2n}$, $n = 2,3,4,5$. These cases can also be analysed simultaneously. Here $dH = 0$, whence $|F|^2 = 0$ and also $|H|^2 = 0$, whence $|d\phi|^2 = 0$. This means that ϕ can only depend on x^-. Therefore $\nabla_+\phi = \nabla_i\phi = 0$. We now insert these into equation (2.4) to obtain the following implications:

$$\begin{aligned}
\nabla_+\nabla_+\phi = 0 &\implies \textstyle\sum_i \mathrm{Tr}(F_{+i})^2 = 0 \implies F_{+i} = 0\\
\nabla_i\nabla_i\phi = 0 &\implies \textstyle\sum_j \mathrm{Tr}(F_{ij})^2 = 0 \implies F_{ij} = 0\\
\nabla_-\nabla_+\phi = 0 &\implies \quad \mathrm{Tr}(F_{+-})^2 = 0 \implies F_{+-} = 0\,.
\end{aligned}$$

As a result the only nonzero component of F is F_{-i} and this means that we can choose a gauge where the only nonzero component of A is A_-. Moreover, because $F_{+-} = 0$, $\partial_+ A_- = 0$. The Yang-Mills equation (2.6) becomes

$$\sum_i \partial_i^2 A_- = 0\,, \tag{6.5}$$

which says that A_- is harmonic in the transverse coordinates. Applying the transverse laplacian to the only nonzero component of the hessian of ϕ

$$\nabla_-\nabla_-\phi = \frac{N}{4}\,\mathrm{Tr}\sum_i(\partial_i A_-)^2$$

we obtain

$$\mathrm{Tr}\sum_{i,j}\left(\partial_i A_- \partial_j^2\partial_i A_- + \partial_i\partial_j A_- \partial_i\partial_j A_-\right) = 0\,.$$

Using equation (6.5), we obtain

$$\mathrm{Tr}\sum_{i,j}\,(\partial_i\partial_j A_-)^2 = 0\,,$$

whence $\partial_i\partial_j A_- = 0$. In other words,

$$A_-(x^-, x^i) = a(x^-) + f_i(x^-)x^i\,,$$

where a and f_i are arbitrary smooth functions of x^-. Finally the dilaton $\phi(x^-)$ is obtained by solving the second order ODE

$$\phi'' = \frac{N}{4} \operatorname{Tr} \sum_i (f_i)^2 \,,$$

which, since the right-hand side is differentiable, has a unique differentiable solution for specified initial conditions at least locally.

8. $\mathbb{R}^{1,2} \times S^7$. Choose coordinates (x^-, x^+, y) for $\mathbb{R}^{1,2}$ and x^α for S^7. As discussed in section 4, there is a coframe $\boldsymbol{\theta}$ relative to which the only nonzero components of F are $\mathring{F}_{-i}$ and $\mathring{F}_{ij}$, the latter in linear combinations belonging to a $\mathfrak{spin}(7)$ subalgebra of $\mathfrak{so}(8)$. Moreover, since $dH \neq 0$ one has that $|F|^2 > 0$ and hence the $\mathring{F}_{ij}$ cannot all vanish.

   The hessian of the dilaton is given by

$$\nabla_\mu \nabla_\nu \phi = \frac{N}{4} \theta_\mu{}^a \theta_\nu{}^b \operatorname{Tr} \sum_i \mathring{F}_{ai} \mathring{F}_{bi} \,,$$

   where, as discussed in section 4, the above trace defines a positive-definite quadratic form. Since the dilaton obeys $\nabla_\alpha \phi = 0$, we see that $\nabla_\alpha \nabla_\alpha \phi = 0$, whence $\theta_\alpha{}^a \mathring{F}_{ai} = 0$ for all α, i. This expands to

$$\theta_\alpha{}^- \mathring{F}_{-i} + \theta_\alpha{}^j \mathring{F}_{ji} = 0 \,. \tag{6.6}$$

   Expanding F relative to this frame, one finds

$$F_{\mu\nu} = (\theta_\mu{}^- \theta_\nu{}^i - \theta_\nu{}^- \theta_\mu{}^i) \mathring{F}_{-i} + \theta_\mu{}^i \theta_\nu{}^j \mathring{F}_{ij} \,;$$

   whence

$$F_{\alpha\nu} = -\theta_\nu{}^- \theta_\alpha{}^i \mathring{F}_{-i}$$

   after using equation (6.6). In particular, since $F_{\alpha\alpha} = 0$, we see that either $\theta_\alpha{}^- = 0$ or $\theta_\alpha{}^i \mathring{F}_{-i} = 0$. Similarly, we have that

$$F_{\alpha\beta} = \theta_\alpha{}^- \theta_\beta{}^i \mathring{F}_{-i} = -\theta_\alpha{}^i \theta_\beta{}^- \mathring{F}_{-i} \,,$$

   whence either if $\theta_\alpha{}^- = 0$ or $\theta_\alpha{}^i \mathring{F}_{-i} = 0$ we see that $F_{\alpha\beta} = 0$. In other words, this means that $\operatorname{Tr} F \wedge F$ cannot have a component in $\Omega^4(S^7)$ contradicting equation (2.2) and the fact that here dH is a nonzero 4-form on S^7.

9. $\mathbb{R}^{1,9}$. Here we can take the coframe $\boldsymbol{\theta}$ to be a coordinate coframe: (dx^-, dx^+, dx^i) and hence $F = \mathring{F}$, which only has components F_{-i}. This means that the Killing spinor, being annihilated by F, obeys $dx^- \varepsilon = 0$. The dilatino equation $d\phi\varepsilon = 0$ implies that $d\phi$ is proportional to dx^-, whence ϕ only depends on x^-.

Since F_{-i} are the only nonzero components, we can choose the gauge $A{=}A_{-}(x^{-}, x^{i}) \times dx^{-}$, whence equation (2.4) becomes

$$\phi'' = \frac{N}{4} \operatorname{Tr} \sum_i (\partial_i A_-)^2 \,. \tag{6.7}$$

The Yang-Mills equation (2.6) is equivalent to

$$\sum_i \partial_i^2 A_- = 0 \,,$$

which says that A_- is harmonic in the transverse space. Applying the transverse laplacian $\sum_i \partial_i^2$ to equation (6.7) and using that A_- is harmonic, we obtain

$$\operatorname{Tr} \sum_{i,j} (\partial_i \partial_j A_-)^2 = 0 \,,$$

which says that the hessian of A_- vanishes; whence A_- is at most linear in the transverse variables:

$$A_- = a(x^-) + f_i(x^-) x^i \,,$$

where a, f_i are arbitrary smooth functions of x^-. Finally we obtain the dilaton by solving the second order ODE

$$\phi'' = \frac{N}{4} \operatorname{Tr} \sum_i (f_i)^2 \,,$$

which has (at least locally) a unique solution for prescribed initial data.

10. $\mathbb{R}^{1,3} \times S^3 \times S^3$, $\mathbb{R}^{1,6} \times S^3$. These two cases can be analysed simultaneously. The geometries are both of the form $\mathbb{R}^{1,n} \times X$, where X is a compact semisimple Lie group. We will choose coordinates $x^\sigma = (x^-, x^+, x^s)$ for the flat factor and x^α for the semisimple Lie group.

    Since $dH = 0$ and hence $|F|^2 = 0$, we know that there is a coframe $\boldsymbol{\theta}$ relative to which equation (4.3) holds and hence that the hessian of the dilaton is given by

    $$\nabla_\mu \nabla_\nu \phi = \frac{N}{4} \theta_\mu{}^- \theta_\nu{}^- \operatorname{Tr} \sum_i \left(\mathring{F}_{-i} \right)^2 \,.$$

    We would like to consider backgrounds with $F \neq 0$, whence the above trace is positive. Since $\nabla_\alpha \phi = 0$, we find that $\theta_\alpha{}^- = 0$. This means that the Killing spinor under consideration is annihilated by Clifford multiplication with $\theta^- = \theta_\sigma{}^- dx^\sigma$ which only has components in the flat factor.

    As we will now argue, this means that we can choose a coordinate coframe (dx^-, dx^+, dx^s) for the flat factor relative to which the Killing spinor is annihilated by dx^-.

    We first observe that Killing spinors in $\mathbb{R}^{1,n} \times X$ are parallel with respect to a product connection (with torsion). Standard facts about holonomy representations can then be invoked to show that parallel spinors are linear combinations of tensor products of parallel spinors in each of the factors (see, for example, [25]). Strictly speaking this is true only when we complexify, since otherwise the spinor representations are not generally tensor products.

Therefore a (complex) Killing spinor in $\mathbb{R}^{1,n} \times X$ can be written as a linear combination

$$\varepsilon = \sum_i \alpha_i \otimes \eta_i \,,$$

where each of the α_i are parallel in $\mathbb{R}^{1,n}$ and the η_i are parallel in X; and where we can assume, without loss of generality, that the η_i are linearly independent. The connection defining parallel transport in $\mathbb{R}^{1,n}$ is the (flat) Levi–Cività connection; whence the α_i are actually constant relative to a coordinate coframe made out of flat coordinates.

Similarly, there is a frame for the Lie group in which the η_i are constant. This can be seen as follows. For a Lie group with a bi-invariant metric there are two canonical flat connections: corresponding to the (metric compatible) absolute parallelisms obtained by left and right multiplication. The corresponding connection one-forms are the right- and left-invariant Maurer–Cartan one-forms, respectively. Let us choose the left-invariant Maurer-Cartan form, for definiteness. Then the corresponding parallel spinor equation is

$$d\eta_i(g) + R(g^{-1}dg)\eta_i(g) = 0\,,$$

where R is the relevant spinorial representation of the Lie algebra and we are letting $g = g(x^\alpha)$ label the points in the group. With some abuse of notation we can let R also stand for the spinorial representation of the Lie group and hence $R(g^{-1}dg) = R(g)^{-1}dR(g)$, whence the above equation can be rewritten as

$$R(g)^{-1}d(R(g)\eta_i(g)) = 0\,,$$

whose solution is manifestly given by $\eta_i(g) = R(g)^{-1}\eta_i(e)$, with e the identity element in the Lie group.

In summary, we have that

$$\varepsilon(x) = \left(\mathbb{1} \otimes R(g)^{-1}\right) \sum_i \alpha_i \otimes \eta_i(e) = \left(\mathbb{1} \otimes R(g)^{-1}\right) \varepsilon_0 \,,$$

where ε_0 is a constant spinor. In other words, there exists a frame and hence a dual coframe, relative to which the Killing spinors are actually constant. Using a constant Lorentz transformation Λ we can take this coframe to $\boldsymbol{\theta} = (\theta^-, \theta^+, \theta^i)$ relative to which $\theta^-\varepsilon_0 = 0$ and hence that the only possibly nonzero components of F are $\mathring{F}_{-i}$. A priori, the Lorentz transformation Λ may not respect the decomposition $\mathbb{R}^{1,n} \times X$; that is, it may not belong to the subgroup $\mathrm{SO}(1,n) \times \mathrm{SO}(9-n) \subset \mathrm{SO}(1,9)$ of the ten-dimensional Lorentz group; however we have seen above that θ^- is a one-form on $\mathbb{R}^{1,n}$, whence Λ can be chosen to belong to this subgroup. Since constant Lorentz transformations preserve flat coordinates, we can choose $\theta^- = dx^-$. In other words, we have that $\theta_\mu{}^- = 0$ except for $\theta_-{}^- = 1$. This means that the only nonzero components of F are those of the form F_{-s} and $F_{-\alpha}$. The hessian equation for the dilaton says the only nonzero element of the hessian is $\nabla_-\nabla_-\phi$.

At this moment the rest of the analysis follows *mutatis mutandis* the case of $\mathrm{CW}_{2n}(\boldsymbol{\lambda}) \times S^3 \times \mathbb{R}^{7-2n}$ discussed above.

6.2 Supersymmetry

We must distinguish between two separate classes of supersymmetric backgrounds with nontrivial gauge fields.

1. Backgrounds with $|H|^2 > 0$. These are the backgrounds with an S^3 factor: $\mathbb{R}^{1,3} \times S^3 \times S^3$, $\mathbb{R}^{1,6} \times S^3$, $\mathrm{CW}_6(\boldsymbol{\lambda}) \times S^3 \times \mathbb{R}$ and $\mathrm{CW}_4(\boldsymbol{\lambda}) \times S^3 \times \mathbb{R}^3$.

   The gaugino variation says that

   $$F\varepsilon = f_I dx^I \wedge dx^- \varepsilon = f_i dx^I \cdot dx^- \varepsilon = 0\,,$$

   which is equivalent, for nonzero F, to $dx^-\varepsilon = 0$.

   On a spinor ε annihilated by dx^-, the dilatino variation says that

   $$\left(Q_I dx^I + \frac{1}{2} H_S\right)\varepsilon = 0\,, \tag{6.8}$$

   where H_S is the three-form for the sphere(s). The same reasoning as in Section 3 allows us to conclude that solutions of (6.8) coincide with the kernel of the projector

   $$\Pi = \frac{1}{2}\mathbb{1} + \frac{1}{4}\frac{H_S \cdot Q}{|Q|^2}\,,$$

   where $Q = Q_I dx^I$ has norm $|Q|^2 = \frac{1}{4}|H_S|^2$, which is therefore nonzero. In summary, Killing spinors are in one-to-one correspondence with the subspace of the chiral spinor representation of $\mathrm{Spin}(1,9)$ consisting of spinors which are annihilated by dx^- and by the above projector Π. This is clearly a 4-dimensional subspace, whence these backgrounds are $\frac{1}{4}$-BPS.

2. Backgrounds with $|H|^2{=}0$. These are the backgrounds without S^3 factors: $\mathrm{CW}_{10}(\boldsymbol{\lambda})$, $\mathrm{CW}_8(\boldsymbol{\lambda}) \times \mathbb{R}^2$, $\mathrm{CW}_6(\boldsymbol{\lambda}) \times \mathbb{R}^4$, $\mathrm{CW}_4(\boldsymbol{\lambda}) \times \mathbb{R}^6$ and $\mathbb{R}^{1,9}$.

   The gaugino variation says that

   $$F\varepsilon = f_i dx^i \wedge dx^- \varepsilon = 0\,,$$

   which is equivalent, for nonzero F, to $dx^-\varepsilon = 0$; and on such spinors, the dilatino equation is automatically satisfied.

   In summary, Killing spinors are in one-to-one correspondence with the subspace of the chiral spinor representation of $\mathrm{Spin}(1,9)$ consisting of spinors which are annihilated by dx^-. This is clearly an 8-dimensional subspace, whence these backgrounds are $\frac{1}{2}$-BPS.

7. On the moduli space of parallelisable backgrounds

The backgrounds in Table 4 each come with moduli and taken in their totality comprise the moduli space of simply-connected parallelisable backgrounds. This moduli space is infinite-dimensional due to the arbitrary functions entering in the expressions for the dilaton and the gauge fields, when nonzero. There are also geometric moduli: the radii of curvature of the AdS_3, S^3 and SU(3) factors, and the eigenvalues $\boldsymbol{\lambda}$ appearing in the definition of $\mathrm{CW}_{2n}(\boldsymbol{\lambda})$. Focussing on the geometric moduli for simplicity, we remark that all the backgrounds are connected by the following geometric limits:

- $\mathrm{SU}(3) \rightsquigarrow \mathbb{R}^8$ by taking the radius of curvature to infinity;
- $S^3 \rightsquigarrow \mathbb{R}^3$ by taking the radius of curvature to infinity;
- $\mathrm{AdS}_3 \rightsquigarrow \mathbb{R}^{1,2}$ by taking the radius of curvature to infinity;
- $\mathrm{CW}_{2n}(\lambda_1, \ldots, \lambda_{n-1}) \rightsquigarrow \mathrm{CW}_{2n-2}(\lambda_1, \ldots, \lambda_{n-2}) \times \mathbb{R}^2$ by taking $\lambda_{n-1} \to 0$;
- $\mathrm{AdS}_3 \times S^3 \times \mathbb{R}^4 \rightsquigarrow \mathrm{CW}_6(\boldsymbol{\lambda}) \times \mathbb{R}^4$ by taking a plane wave limit [26, 27]; and
- $\mathrm{AdS}_3 \times S^3 \times S^3 \times \mathbb{R} \rightsquigarrow \mathrm{CW}_8(\boldsymbol{\lambda}) \times \mathbb{R}^2$ again by a plane wave limit.

Some of these plane wave limits have appeared in [28, 29]. As the above geometries are parallelised Lie groups, these plane wave limits can also be understood as group contractions [30, 31] in the sense of Inönü and Wigner [32].

Finally we should remark that we have classified the simply-connected backgrounds. Equivalently we do not distinguish between backgrounds which are locally isometric; for example, two backgrounds which are obtained as different discrete quotients of the same simply-connected background. More generally, in order to classify all smooth backgrounds, one must quotient the simply-connected geometries in Table 4 by all possible freely-acting discrete subgroups of symmetries which preserve some supersymmetry and are free of singularities. This is a much more delicate problem and we will only mention the fact that performing this quotient corresponds, at the level of the conformal field theory, to an orbifold construction, whence the string theory remains, in principle, exactly solvable.

Acknowledgments

We are grateful to Shyamoli Chaudhuri, Yosuke Imamura, Tomás Ortín and Yuji Tachikawa for helpful correspondence and discussions. This work was completed while J.M.F. was visiting CERN, whom he would like to thank for support and hospitality, and T.K. and S.Y. were participating at the Summer Institute “Fuji 2003” in Shizuoka, Japan, whose organisers they would like to thank for hospitality. The research of J.M.F. is partially funded by the UK EPSRC grant GR/R62694/01 and that of S.Y. by a JSPS Research Fellowship for Young Scientists. J.M.F. is also a member of EDGE, Research Training Network HPRN-CT-2000-00101, supported by The European Human Potential Programme.

A. Clifford algebra conventions

Our Clifford algebra conventions mostly follow the book [33], but we will review them here briefly.

Let $\mathbb{R}^{t,s}$ denote the real $(t+s)$-dimensional vector space with inner product obtained from the norm

$$|\boldsymbol{x}|^2 = -(x^1)^2 - \cdots - (x^t)^2 + (x^{t+1})^2 + \cdots + (x^{t+s})^2 ,$$

for $\boldsymbol{x} = (x^1, \ldots, x^{s+t}) \in \mathbb{R}^{t,s}$. By definition the real Clifford algebra $\mathrm{C}\ell(t,s)$ is generated by $\mathbb{R}^{t,s}$ (and the identity $\mathbb{1}$) subject to the Clifford relation

$$\boldsymbol{x} \cdot \boldsymbol{x} = -|\boldsymbol{x}|^2 \mathbb{1} ,$$

where we ask the reader to pay close attention to the sign! Let $\boldsymbol{e}_a \in \mathbb{R}^{t,s}$ be the elementary basis vectors relative to which $\boldsymbol{x} = \sum x^a \boldsymbol{e}_a$ and let Γ_a denote their image in $\mathrm{C}\ell(t,s)$. Then the Clifford relations become

$$\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = -2\eta_{ab} \mathbb{1} ,$$

where $\eta_{ab} = \langle \boldsymbol{e}_a, \boldsymbol{e}_b \rangle$ are the components of the inner product relative to this basis.

We are interested in ten-dimensional lorentzian signature: $\mathbb{R}^{1,9}$ with orthonormal basis $\boldsymbol{e}_0, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_9$ with $|\boldsymbol{e}_0|^2 = -1$ and $|\boldsymbol{e}_i|^2 = 1$ for $i = 1, \ldots, 9$. As a real associative algebra, $\mathrm{C}\ell(1,9)$ is isomorphic to the algebra of 32×32 real matrices. This is a simple algebra and hence has a unique irreducible representation $\mathcal{S}$, which is real and thirty-two dimensional, the so-called Majorana spinors. The chirality operator $\Gamma_{11} := \Gamma_0 \Gamma_1 \cdots \Gamma_9$ squares to $+\mathbb{1}$ and anticommutes with all Γ_a, whence it commutes with the generators $\frac{1}{2}\Gamma_{ab}$ of $\mathfrak{so}(1,9)$. Therefore $\mathcal{S}$ breaks up into two real sixteen-dimensional irreducible representations of the spin group $\mathrm{Spin}(1,9)$: $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$, the so-called Majorana–Weyl spinors.

The Clifford algebra $\mathrm{C}\ell(1,9)$ is isomorphic as a real vector space (but *not* as an algebra) to the exterior algebra $\Lambda\mathbb{R}^{1,9}$, whose elements are real linear combinations of monomials of the form $\boldsymbol{e}_{a_1} \wedge \boldsymbol{e}_{a_2} \wedge \cdots \wedge \boldsymbol{e}_{a_k}$. There is a natural isomorphism $\Lambda\mathbb{R}^{1,9} \to \mathrm{C}\ell(1,9)$ given by sending

$$\boldsymbol{e}_{a_1} \wedge \boldsymbol{e}_{a_2} \wedge \cdots \wedge \boldsymbol{e}_{a_k} \mapsto \Gamma_{a_1 a_2 \cdots a_k} . \tag{A.1}$$

This map also preserves the canonical $\mathbb{Z}_2$ gradings of $\mathrm{C}\ell(1,9)$ and of $\Lambda\mathbb{R}^{1,9} = \Lambda^{\mathrm{even}}\mathbb{R}^{1,9} \oplus \Lambda^{\mathrm{odd}}\mathbb{R}^{1,9}$. In this way, elements of $\Lambda\mathbb{R}^{1,9}$ can act on $\mathcal{S}$: even elements preserving the chirality, whence mapping $\mathcal{S}_\pm \to \mathcal{S}_\pm$, and odd elements reversing it, whence mapping $\mathcal{S}_\pm \to \mathcal{S}_\mp$.

Now let (M, g) be a lorentzian ten-dimensional manifold. We can choose local orthonormal frames for the tangent bundle TM and dual coframes for the cotangent bundle T^*M. Relative to such a coframe, each cotangent space is isomorphic to $\mathbb{R}^{1,9}$ as an inner product space and we can construct at each point a Clifford algebra $\mathrm{C}\ell(1,9)$. As we let the point vary, these algebras patch up nicely to yield a bundle $\mathrm{C}\ell(T^*M)$ of Clifford algebras which, as a vector bundle, is isomorphic to ΛT^*M. The isomorphism (A.1) also extends to give a map $\Lambda T^*M \to \mathrm{C}\ell(T^*M)$.

JHEP10(2003)012

If in addition, (M, g) is spin, then there is a (not necessarily unique) vector bundle S associated to the irreducible representation $\mathcal{S}$ of $C\ell(1,9)$. Furthermore this bundle breaks up into sub-bundles $S = S_+ \oplus S_-$ corresponding to the irreducible representations of the spin group. Sections of ΛT^*M — that is, differential forms — act naturally on sections of S via the isomorphism $\Lambda T^*M \to C\ell(T^*M)$ and the natural pointwise action of $C\ell(T^*M)$ on S. This action extends to an action of forms with values in a vector bundle V which maps sections of S to sections of $S \otimes V$. In our case, V will be the adjoint bundle of the gauge bundle. This now explains what we mean by the action of forms on spinors, as in equation (2.7).

Several times during the calculations in this paper we have come across Clifford squares; that is, the repeated action of a differential form on a spinor. For α a 1-form, we obtain simply

$$\alpha^2 = -|\alpha|^2 \mathbb{1}\,;$$

whereas for ω a 2-form, we find

$$\omega^2 = \omega \wedge \omega - |\omega|^2 \mathbb{1}\,.$$

For H a 3-form we find

$$H^2 = -\frac{1}{4} H_{abm} H^m{}_{cd} \Gamma^{abcd} + |H|^2 \mathbb{1}\,,$$

which, if H satisfies the Jacobi identity, simplifies to

$$H^2 = |H|^2 \mathbb{1}\,.$$

Finally, if Θ is a 4-form,

$$\Theta^2 = \Theta \wedge \Theta - \frac{1}{8} \Theta_{abmn} \Theta^{mn}{}_{cd} \Gamma^{abcd} + |\Theta|^2 \mathbb{1}\,.$$

References

[1] D. Sadri and M.M. Sheikh-Jabbari, *String theory on parallelizable PP-waves*, *J. High Energy Phys.* **06** (2003) 005 [hep-th/0304169].

[2] J. Figueroa-O'Farrill, *On parallelisable NS-NS backgrounds*, *Class. and Quant. Grav.* **20** (2003) 3327 [hep-th/0305079].

[3] T. Kawano and S. Yamaguchi, *Dilatonic parallelizable NS-NS backgrounds*, *Phys. Lett.* **B 568** (2003) 78 [hep-th/0306038].

[4] C.G. Callan Jr., J.A. Harvey and A. Strominger, *Supersymmetric string solitons*, hep-th/9112030.

[5] E. Witten, *Nonabelian bosonization in two dimensions*, *Commun. Math. Phys.* **92** (1984) 455.

[6] G. Papadopoulos, J.G. Russo and A.A. Tseytlin, *Solvable model of strings in a time-dependent plane-wave background*, *Class. and Quant. Grav.* **20** (2003) 969 [hep-th/0211289].

[7] M. Blau and M. O'Loughlin, *Homogeneous plane waves*, *Nucl. Phys.* **B 654** (2003) 135 [hep-th/0212135].

[8] D.J. Gross, J.A. Harvey, E.J. Martinec and R. Rohm, *The heterotic string*, *Phys. Rev. Lett.* **54** (1985) 502.

[9] A.H. Chamseddine, *$N = 4$ supergravity coupled to $N = 4$ matter*, *Nucl. Phys.* **B 185** (1981) 403.

[10] L. Brink, J.H. Schwarz and J. Scherk, *Supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 121** (1977) 77.

[11] G.F. Chapline and N.S. Manton, *Unification of Yang-Mills theory and supergravity in ten-dimensions*, *Phys. Lett.* **B 120** (1983) 105.

[12] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, *Ten-dimensional Maxwell-Einstein supergravity, its currents and the issue of its auxiliary fields*, *Nucl. Phys.* **B 195** (1982) 97.

[13] É. Cartan and J. Schouten, *On the geometry of the group manifold of simple and semisimple groups*, *Nederl. Akad. Wetensch. Proc. Ser. A* **29** (1926) 803.

[14] É. Cartan and J. Schouten, *On riemannian geometries admitting an absolute parallelism*, *Nederl. Akad. Wetensch. Proc. Ser. A* **29** (1926) 933.

[15] J. Wolf, *On the geometry and classification of absolute parallelisms, I*, *J. Diff. Geom.* **6** (1971) 317.

[16] J. Wolf, *On the geometry and classification of absolute parallelisms, II*, *J. Diff. Geom.* **7** (1972) 19.

[17] M. Cahen and M. Parker, *Parallélismes absolus des variétés lorentziennes*, *Ann. Inst. Fourier (Grenoble)* **27** (1977), no. 1, xi, 251.

[18] A. Medina and P. Revoy, *Algèbres de Lie et produit scalaire invariant*, *Ann. Scient. Éc. Norm. Sup.* **18** (1985) 553.

[19] J.M. Figueroa-O'Farrill and S. Stanciu, *On the structure of symmetric selfdual Lie algebras*, *J. Math. Phys.* **37** (1996) 4121 [hep-th/9506152].

[20] M. Cahen and N. Wallach, *Lorentzian symmetric spaces*, *Bull. Am. Math. Soc.* **76** (1970) 585.

[21] R. Bryant, *Pseudo-riemannian metrics with parallel spinor fields and vanishing Ricci tensor*, in *Global analysis and harmonic analysis*, Marseille-Luminy, 1999, J. Bourgignon, T. Branson and O. Hijazi eds., vol. 4 of *Sémin. Congr.*, pp. 53–94 Soc. Math. France, Paris 2000 [math.DG/0004073].

[22] J.M. Figueroa-O'Farrill, *Breaking the M-waves*, *Class. and Quant. Grav.* **17** (2000) 2925 [hep-th/9904124].

[23] R. Khuri, *Remarks on string solitons*, *Phys. Rev.* **D 48** (1993) 2947 [hep-th/9305143].

[24] M.J. Duff and J.X. Lu, *Elementary five-brane solutions of $D = 10$ supergravity*, *Nucl. Phys.* **B 354** (1991) 141.

[25] T. Leistner, *Lorentzian manifolds with special holonomy and parallel spinors*, in *Proceedings of the 21st Winter School Geometry and Physics*, Srní, 2001, no. 69, 2002, pp. 131–159.

[26] R. Penrose, *Any space-time has a plane wave as a limit*, in *Differential geometry and relativity*, Reidel, Dordrecht 1976, pp. 271–275.

[27] R. Güven, *Plane wave limits and T-duality*, *Phys. Lett.* **B 482** (2000) 255 [hep-th/0005061].

[28] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, *Class. and Quant. Grav.* **19** (2002) L87 [hep-th/0201081].

[29] M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos, *Penrose limits, supergravity and brane dynamics*, *Class. and Quant. Grav.* **19** (2002) 4753 [hep-th/0202111].

[30] D.I. Olive, E. Rabinovici and A. Schwimmer, *A class of string backgrounds as a semiclassical limit of WZW models*, *Phys. Lett.* **B 321** (1994) 361 [hep-th/9311081].

[31] S. Stanciu and J. Figueroa-O'Farrill, *Penrose limits of Lie branes and a Nappi-Witten braneworld*, *J. High Energy Phys.* **06** (2003) 025 [hep-th/0303212].

[32] E. Inönü and E. Wigner, *On the contraction of groups and their representations*, *Proc. Nat. Acad. Sci. USA* **39** (1956) 510.

[33] H. Lawson and M. Michelsohn, *Spin geometry* Princeton University Press, Princeton 1989.

JHEP10(2003)012